Int. J. Heat Mass Transfer. Vol. 22, pp. 167–170
0017-9310/79/0101-0167 $02.00/0

SHORTER COMMUNICATIONS

SOME IMPROVEMENTS TO THE SOLUTION OF STEFAN-LIKE PROBLEMS

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(*Received* 14 *February* 1978 *and in revised form* 10 *May* 1978)

NOMENCLATURE

a, b, c, arbitrary coefficients, equation (13);
C_p, specific heat;
h, convective heat-transfer coefficient;
k, thermal conductivity;
L, latent heat of fusion;
Q, arbitrary heat flux at the crust–liquid interface;
t, time;
T, temperature;
T_0, reference temperature;
x, distance measured from the wall–crust interface.

Greek symbols

α, thermal diffusivity;
γ, arbitrary coefficient, equation (15);
δ, instantaneous frozen crust thickness;
Δ, dimensionless crust thickness, equation (6);
θ_l, dimensionless wall–crust interface temperature, equation (6);
θ_s, dimensionless temperature within the frozen crust, equation (6);
μ, arbitrary coefficient, equation (28);
ε, inverse Stefan number for freezing, equation (6);
ξ, dimensionless coordinate, equation (6);
λ, overall freezing coefficient, equation (21);
ρ, density;
τ, dimensionless time, equation (6).

Subscripts

av, the upper and lower bounds averaging method;
b, bulk;
f, fusion;
LB, lower bound;
R, the refined integral heat balance method;
s, solidified crust;
UB, upper bound;
w, wall.

INTRODUCTION

TWO APPROXIMATE analytical methods are developed for solving one-dimensional transient heat-conduction problems with phase transformation, where the growth rate of a frozen crust (layer) on a cold wall is sought. The first method involves a refining of the integral-heat-balance (IHB), as introduced by Goodman [1], by carrying out a double space integration in a manner similar to that suggested by Volkov and Li-Orlov [2]. Since this technique [2] was found to yield an appreciable improvement in accuracy of the IHB [1], as applied to unsteady heat conduction problems without phase transformation; such a technique is examined here for Stefan-like problems. In addition, the solidification bounds averaging method, as proposed by Hamil and Bankoff [3], is reexamined by arithmetically averaging the upper and lower bounds for the solidification interface rather than averaging the denominator of an expression for the interface position as formulated in [3]. Such a method has the utility of assessing the frozen layer thickness independent of a knowledge of the temperature field in the crust; yielding an improved accuracy over the previous formulation [3], particularly for higher values of the Stefan number.

To assess the accuracy of the present formulations, analytic expressions for the instantaneous position of the solidification front, $\delta(t)$ (i.e. frozen layer thickness) are presented and compared with the known exact solution for the freezing of a stagnant liquid on an isothermal wall [4], commonly referred to as the Neumann problem. Such a comparison illustrates that the integration techniques suggested here for Stefan-like problems offers an improvement in accuracy above previous formulations [1, 3].

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ANALYSIS

The generalized system to be analyzed is illustrated in Fig. 1. A constant heat flux (Q) is defined at the solid–liquid (S–L) interface as well as an arbitrary time dependent condition at the wall surface ($x = 0$). For the case where a convective heat transfer at the (S–L) interface is involved, $Q = h(T_b - T_f)$.

The transient heat conduction equation within the frozen layer is

$$\alpha_s \frac{\partial^2 T_s}{\partial x^2} = \frac{\partial T_s}{\partial t}. \tag{1}$$

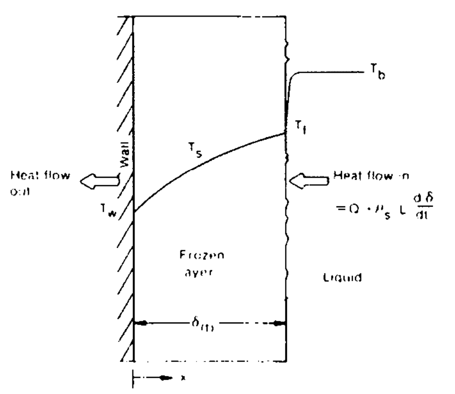

FIG. 1. Schematic diagram of transient freezing onto a cold wall.

Initially the frozen layer thickness is zero, i.e.,

$$\delta(0) = 0, \tag{2}$$

and the boundary conditions for this frozen layer can be written as:

$$T_s(0, t) = T_w(t), \tag{3}$$

$$T_s(\delta, \tau) = T_f. \tag{4}$$

To define the instantaneous thickness of the frozen layer, $\delta(t)$, an additional equation must be written for the heat balance at this moving boundary; thus,

$$k_s \frac{\partial T_s}{\partial x}(\delta, t) = Q + \rho_s L \frac{d\delta}{dt}. \tag{5}$$

Constant but different thermalphysical properties for the liquid and solid phases are assumed.

The analysis can be simplified by introducing the following set of transformations:

$$\xi \equiv \frac{x}{\delta(t)} \quad \text{(Dimensionless coordinate)},$$

$$\tau \equiv \frac{Q^2}{k_s^2(T_f - T_0)^2}\alpha_s t \quad \text{(Dimensionless time)},$$

$$\Delta \equiv \frac{Q}{k_s(T_f - T_0)}\delta(t)$$

$$\text{(Dimensionless crust thickness)}, \tag{6}$$

$$\varepsilon \equiv \frac{L}{Cp_s(T_f - T_0)}$$

(Inverse Stefan number for freezing),

$$\theta_s(\xi, \tau) \equiv \frac{(T_s - T_0)}{(T_f - T_0)}$$

(Dimensionless temperature),

$$\theta_I(\tau) \equiv \frac{(T_w - T_0)}{(T_f - T_0)}$$

(Dimensionless interface temperature).

Equations (1)–(5) then reduce to

$$\Delta^2 \frac{\partial \theta_s}{\partial \tau} = \frac{\partial^2 \theta_s}{\partial \xi^2} + \frac{\xi}{2}\frac{d\Delta^2}{d\tau}\frac{\partial \theta_s}{\partial \xi}, \tag{7}$$

$$\Delta(\tau = 0) = 0, \tag{8}$$

$$\theta_s(0, \tau) = \theta_I(\tau), \tag{9}$$

$$\theta_s(1, \tau) = 1.0, \tag{10}$$

$$\frac{\partial \theta_s}{\partial \xi}(1, \tau) = \Delta + \frac{\varepsilon}{2}\frac{d\Delta^2}{d\tau}. \tag{11}$$

Upon integrating equation (7) twice with respect to space, making use of the boundary conditions given by equations (9)–(11), and once with respect to time, knowing that $\Delta(\tau = 0) = 0$, we obtain the following integral form for the dimensionless crust thickness

$$\Delta^2 = \frac{\int_0^\tau [(\theta_I - 1) + \Delta]\, d\tau}{\int_0^1 \eta\theta_s\, d\eta - \frac{1}{2}(\varepsilon + 1)}. \tag{12}$$

Equation (12) forms the basis upon which the two methods suggested here are investigated.

The refined integral heat balance (RIHB)

Unlike the integration sequence outlined above, Goodman [1] originally carried out a heat flow balance on a control volume defined by the penetration distance of the moving interface, $\delta(t)$, where the starting heat equation is integrated once with respect to space. The heat fluxes at the boundaries were then evaluated from an assumed polynomial for the temperature field that satisfies the boundary conditions of the problem. An expression was then obtained for the instantaneous interface velocity, $d\delta(t)/dt$, which upon integration with respect to time gave $\delta(t)$. The solution obtained in this manner is found to be relatively sensitive to small variations in the Kernel function from the exact solution. The accuracy however is relatively independent of the order of the assumed polynomial.

Following the integration sequence arguments of Volkov and Li-Orlov [2] (which however were formulated for nonlinear transient heat-conduction problems, where the nonlinearity is due to temperature dependent properties) the starting equation for the frozen crust (equation 7) can be integrated immediately resulting in equation (12), which expresses the moving solidification front as an integral function of the boundary conditions and temperature field in the frozen layer [i.e. $\theta_s(\xi, \tau)$ which gives $T_s(x, t)$]. Assuming a second degree polynomial in space and time for the temperature field in the frozen layer of the form:

$$\theta_s(\xi, \tau) = a - b(1 - \xi) + c(1 - \xi)^2. \tag{13}$$

$\theta_s(\xi, \tau)$ can be expressed in terms of the frozen crust thickness as:

$$\theta_s(\xi, \tau) = 1 + \frac{\gamma}{2}(\Delta - 2\varepsilon)(1 - \xi) - \left[\frac{\gamma}{2}(\Delta - 2\varepsilon) + (1 - \theta_I)\right](1 - \xi)^2, \tag{14}$$

where

$$\gamma = -1 + \left[1 + \frac{8\varepsilon(1 - \theta_I)}{(\Delta - 2\varepsilon)^2}\right]^{1/2}. \tag{15}$$

Equation (14) satisfies the boundary conditions given by the equations (9)–(11). From the equation (14), the denominator of equation (12) is found to be

$$\int_0^1 \eta\theta_s\, d\eta - \frac{1}{2}(\varepsilon + 1) = -\frac{1}{12}\left[6\varepsilon + (1 - \theta_I) - \frac{\gamma}{2}(\Delta - 2\varepsilon)\right], \tag{16}$$

and the overall expression for $\Delta(\tau)$ (equation 12) becomes

$$\Delta^2(\tau) = \frac{12\int_0^\tau [(1 - \theta_I) - \Delta]\, d\tau}{[6\varepsilon + (1 - \theta_I) - \gamma/2(\Delta - 2\varepsilon)]}. \tag{17}$$

It is easy to show that equation (17) holds also for melting problems, with the sign of L reversed in the definition of the dimensionless parameter ε. As will be shown such an expression is relatively insensitive to small variations in the Kernel function from the exact solution and results in a more accurate prediction than the original formulation of the IHB method [1].

The upper and lower bounds averaging method

The bounds averaging method provides a simplified expression for the instantaneous frozen layer thickness that forms on a cold surface, independent of knowing the temperature distribution within the frozen crust. In the present analysis this is accomplished by taking the arithmetic average of the upper and lower bounds of the moving interface position, where the dimensionless temperature (θ_s) in the denominator of equation (12) is taken as equal to 1 (corresponding to a flat temperature profile across the frozen layer equal to the fusion temperature) and $\theta_s = \theta_I$ (corresponding to a flat temperature profile across the frozen layer equal to the wall temperature), respectively.

For $\theta_I \leqslant \theta_s \leqslant 1$, the upper and lower bounds of the integral in the denominator of equation (12) are

$$\tfrac{1}{2}\theta_I \leqslant \int_0^1 \eta\theta_s \, d\eta \leqslant \tfrac{1}{2}. \quad (18)$$

Substituting the above inequality into equation (12), expressions for the upper and lower bounds of Δ^2 are obtained as

$$2\frac{\int_0^\tau (1-\theta_I-\Delta)\,d\tau}{(1-\theta_I+\varepsilon)} \leqslant \Delta^2 \leqslant 2\frac{\int_0^\tau (1-\theta_I+\Delta)\,d\tau}{\varepsilon}. \quad (19)$$

Up to this point, the solution technique, with respect to the averaging method, is the same as developed by Hamill and Bankoff [3]. In their development, an expression for the moving interface position, $\Delta(\tau)$, is obtained by averaging only the denominators of the upper and lower bounds given in equation (19). However, $\Delta(\tau)$ can also be obtained by taking the arithmetic average of the entire expression, given in equation (19), for the upper and lower bounds of the moving interface position. Thus, the dimensionless crust thickness can be expressed as

$$\Delta(\tau) = 2\lambda(\tau)(\tau)^{1/2}, \quad (20)$$

where

$\lambda(\tau)$ = the overall freezing coefficient

$$= \tfrac{1}{2}[\lambda_{LB}+\lambda_{UB}]. \quad (21)$$

The lower bound freezing coefficient, λ_{LB}, being

$$\lambda_{LB} = \left[\frac{\int_0^\tau (1-\theta_I-\Delta)\,d\tau}{2\tau(1-\theta_I+\varepsilon)}\right]^{1/2}, \quad (22)$$

and the upper bound freezing coefficient, λ_{UB}, being

$$\lambda_{UB} = \left[\frac{\int_0^\tau (1-\theta_I-\Delta)\,d\tau}{2\tau\varepsilon}\right]^{1/2}. \quad (23)$$

Rearranging equation (20), we obtain

$$\Delta^2(\tau) = \left\{\frac{(1-\theta_I+2\varepsilon)+2[\varepsilon(1-\theta_I+\varepsilon)]^{1/2}}{2\varepsilon(1-\theta_I+\varepsilon)}\right\} \times \int_0^\tau (1-\theta_I-\Delta)\,d\tau. \quad (24)$$

Equation (24) gives an implicit expression for the instantaneous frozen layer thickness, $\Delta(\tau)$, in terms of the boundary conditions at the wall–frozen layer and frozen layer–liquid interfaces.

As can be seen from equation (21), the freezing coefficient $\lambda(\tau)$, given by this averaging method, is strongly dependent on Stefan number for freezing $(1/\varepsilon)$ and the boundary conditions at the interfaces but insensitive to the temperature distribution within the frozen layer. This indicates that problems having constant boundary conditions, either in terms of temperature or heat flux, should result in an accurate prediction using the averaging technique.

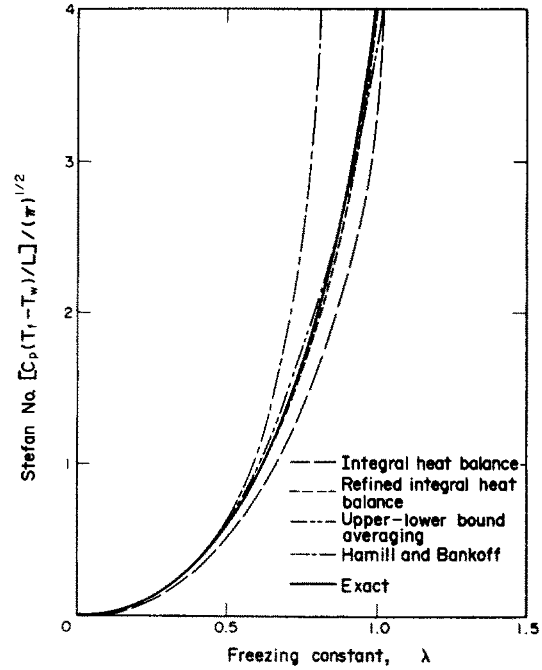

FIG. 2. The freezing constant, λ, of a stagnant liquid at its fusion temperature onto an isothermal cold wall as given by the exact and the approximate solutions.

To assess the accuracy of both the refined integral heat balance and bounds averaging method, as developed here, a comparison is made with the known exact solution to the classical Neumann problem.

ASSESSMENT OF ACCURACY

Considering the freezing of a stagnant liquid at its fusion temperature [i.e. Q being zero] onto an isothermal cold wall [i.e. $T_s(0,t) = T_0$], the two methods developed in the preceding section yield the following expressions.

(a) *The refined integral heat balance method*

For $\theta_I(\tau) = 0$, the refined integral heat balance method (equation 17) yields the following expression for the instantaneous frozen layer thickness

$$\Delta(\tau) = 2\lambda_R(\tau)^{1/2}, \quad (25)$$

Table 1. Freezing constant, λ, for a warm liquid at its fusion temperature on an isothermal wall

Method	Freezing constant λ
I. Integral heat balance [1], (INB)	$\left\{\frac{3[1-(1+\mu)^{1/2}+\mu]}{5+(1+\mu)^{1/2}+\mu}\right\}^{1/2}$
II. Refined integral heat balance (RIHB)	$\left\{\frac{3}{1+\varepsilon[5+(1+\mu)^{1/2}]}\right\}^{1/2}$
III. Upper–lower bounds averaging	$\left\{\frac{(1+2\varepsilon)+2[\varepsilon(1+\varepsilon)]^{1/2}}{8\varepsilon(1+\varepsilon)}\right\}^{1/2}$
IV. Hamill and Bankoff [3]	$\left(\frac{1}{1+2\varepsilon}\right)^{1/2}$
V. Exact solution [4]	$\lambda e^{\lambda^2} \operatorname{erf}\lambda = [\varepsilon(\pi)^{1/2}]^{-1}$

where

$$\lambda_R = \left\{\frac{3}{[1+\varepsilon(6+\gamma_R)]}\right\}^{1/2}, \quad (26)$$

$$\gamma_R = -1+(1+\mu)^{1/2}, \quad (27)$$

$$\mu = 2/\varepsilon. \quad (28)$$

(b) *The upper and lower bounds averaging method*

By substituting $\theta_I(\tau) = 0$ [i.e. $T_s(x = 0, t) = T_0$] into equation (24) and noting that for $Q = 0$, $\Delta(\tau)$ on the RHS of equation (24) is zero, while $\Delta^2(\tau)/\tau = \delta^2(t)/\alpha_s t$. The averaging method thus gives the following expression for the instantaneous frozen layer thickness

$$\Delta(\tau) = 2\lambda_{av}(\tau)^{1/2}, \quad (29)$$

where

$$\lambda_{av} = \left[\frac{(1+2\varepsilon)+2\varepsilon(1+\varepsilon)}{8\varepsilon(1+\varepsilon)}\right]^{1/2}. \quad (30)$$

A listing is shown in Table 1 of the resultant expressions for the freezing constant, λ, as obtained by various approximate methods and the exact solution, while a plot of λ vs the Stefan number is presented in Fig. 2. It is noted that the integration of the heat-conduction equation twice with respect to space, prior to assuming a polynomial for the temperature distribution, yields a significant improvement over the integral heat balance method as applied by Goodman [1] for freezing on an isothermal wall. Such results are in agreement with that of Megerlin [5], who has noted that Goodman's method [1] does not yield very accurate results in problems of freezing and melting. The refined integral heat balance predictions are accurate to within about 0.3% of the exact solution while the integral heat balance is accurate to within about 6%.

For the problem at hand, the averaging method introduced here yields an error no greater than 2%. However, the advantage of this method is that an accurate prediction of the freezing constant can be obtained independent of a knowledge of the temperature distribution within the frozen layer. It is also noted that averaging the denominator of the upper and lower bounds, as introduced by Hamill and Bankoff [3], results in a somewhat less accurate prediction of the freezing constant in this case, especially for higher values of the Stefan number.

As indicated below the methods developed here can be easily applied to other problems, for example the case where the wall temperature is a time dependent function, an illustrative example being

$$\theta_I(\tau) = a\,e^{\phi\tau}, \quad (31)$$

where ϕ is a constant either positive or negative.

The refined integral heat balance gives

$$\Delta(\tau) = 2\lambda_R(\tau)(\tau)^{1/2}, \quad (32)$$

where

$$\lambda_R(\tau) = \left\{\frac{3\left[1-\dfrac{\theta_I(\tau)}{\phi\tau}\right]}{6\varepsilon+(1-\theta_I)+\gamma_R\varepsilon}\right\}^{1/2}. \quad (33)$$

The averaging method in this case results in

$$\Delta(\tau) = 2\lambda_{av}(\tau)(\tau)^{1/2}, \quad (34)$$

where

$$\lambda_{av}(\tau) = \left(\frac{\{(1-\theta_I+2\varepsilon)+2[\varepsilon(1-\theta_I+\varepsilon)]^{1/2}\}\left[1-\dfrac{\theta_I(\tau)}{\phi\tau}\right]}{8\varepsilon(1-\theta_I+\varepsilon)}\right)^{1/2}. \quad (35)$$

Other solutions using the method developed in this work can be found in [6].

CONCLUSION

The two methods presented in this paper provide an accurate prediction of the instantaneous position of the moving boundary as applied to one-dimensional melting and freezing problems. The methods developed can be applied to various problems involving a change-of-phase, with or without a convective boundary condition at the moving front. The important feature of these methods is that the instantaneous frozen layer thickness is given in an integral form which is relatively insensitive to small variations in the Kernel function from the exact solution.

It is noted that the double integration of the heat conduction equation twice with respect to space or the arithmetic averaging of the upper and lower bounds of the freezing (or melting) front position provides an accurate prediction concerning the instantaneous position of the moving boundary. However, the refined integral heat balance technique gives the more accurate and simplified solution to the problems presented compared to the solutions obtained by the bounds averaging method. Results of both methods, however, are in good agreement with the known exact solution to the Neumann problem and illustrate that such methods offer improvement in accuracy above known approximate analytical methods of other investigators [1, 3]. Thus, future work might include application of such methods to the more difficult problems of simultaneous melting and freezing of coupled multimedia and finite geometry systems.

Acknowledgements—This work was supported by the U.S. Nuclear Regulatory Commission, Division of Fast Reactor Safety Research.

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